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Zeta functions and complexities of a semiregular bipartite graph and its line graph ☆

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Abstract

We treat zeta functions and complexities of semiregular bipartite graphs. Furthermore, we give formulas for zeta function and the complexity of a line graph of a semiregular bipartite graph. As a corollary, we present the complexity of a line graph of a complete bipartite graph.

Keywords: Zeta function; Complexity; Semiregular graph; Line graph

1. Introduction

Graphs and digraphs treated here are finite. Let G be a connected graph with a set $V(G)$ of vertices and a set $E(G)$ of edges, and let D be the symmetric digraph corresponding to G. A *path P of length n* in D(or G) is a sequence $P=(v_0, v_1, \ldots, v_{n-1}, v_n)$ of $n+1$ vertices such that consecutive vertices share an arc (or edge) (we do not require that all vertices are distinct). Also, P is called a *(v_0, v_n)-path*. A (v, w)-path is called a *cycle* (or *closed path*) if $v=w$. The *inverse cycle* of a cycle $C=(v, v_1, \ldots, v_{n-1}, v)$ is the cycle $C^{-1}=(v, v_{n-1}, \ldots, v_1, v)$.

We introduce an equivalence relation between cycles. Such cycles $C_1=(v_1, \ldots, v_m)$ and $C_2=(w_1, \ldots, w_m)$ are called *equivalent* if $w_j=v_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. We say that a path has a *backtracking* if a subsequence of the form $\ldots, x, y, x, \ldots$ appears. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle.

The (*Ihara*) *zeta function* of a graph G is defined to be a function of $u \in \mathbf{C}$ with $|u|$ sufficiently small, by

$$\mathbf{Z}(G, u)=\mathbf{Z}_G(u)=\prod_{[C]}(1-u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $|C|$ is the length of C.

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Let G be a connected graph with n vertices $v_1, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}(G)=(a_{ij})$ is the square matrix such that $a_{ij}=|\{e \in E(G) | e=v_i v_j\}|$ if v_i and v_j are adjacent, and $a_{ij}=0$ otherwise. The *degree* $\deg_G v = \deg v$ of a vertex v in G is the number of edges which are adjacent to v. Let $\mathbf{D}=(d_{ij})$ be the diagonal matrix with $d_{ii}=deg_G\ v_i$, and $\mathbf{Q}=\mathbf{D}-\mathbf{I}$.

Ihara [4] defined the zeta function of a regular graph. Hashimoto [3] treated multivariable zeta functions of bipartite graphs. Bass [1] generalized Ihara's result on zeta functions of regular graphs to irregular graphs.

Theorem 1 (*Bass [1]*). *The reciprocal of the zeta function of G is given by*

$$\mathbf{Z}_G(u)^{-1}=(1-u^2)^{l-n}\det(\mathbf{I}-u\mathbf{A}(G)+u^2\mathbf{Q}),$$

where $n=|V(G)|$ and $l=|E(G)|$.

Stark and Terras [7] gave an elementary proof of Theorem 1.1, and discussed three different zeta functions of any graph. Recently, another proof of Theorem 1, and is obtained by Kotani and Sunada [5].

The *complexity* $\kappa(G)$(= the number of spanning trees in G) of a connected graph G is closely related to the zeta function of G. Northshield [6] showed that the complexity of G is given by the derivative of a determinant contained in the reciprocal of its zeta function. For a connected graph G, let $f_G(u)=\det(\mathbf{I}-u\mathbf{A}(G)+u^2\mathbf{Q})$.

Theorem 2 (*Northshield [6]*). *The complexity of G is given as follows*:

$$f'_G(1)=2(l-n)\kappa(G),$$

where $n=|V(G)|$ and $l=|E(G)|$.

The complexities for various graphs were given in [2]. Let $\Phi(G;\lambda)=\det(\lambda\mathbf{I}-\mathbf{A}(G))$ be the *characteristic polynomial* of G. A graph G is called *k-regular* if $\deg_G v=k$ for each vertex $v \in V(G)$. The complexity $\kappa(G)$ of an r-regular graph G with n vertices is

$$\kappa(G)=\frac{1}{n}\Phi'(G;r),$$

where $\Phi'(G;\lambda)$ is the derivative of $\Phi(G;\lambda)$.

The line graph $L(G)$ of a graph G is the graph whose vertices are the edges of G connected (by an edge in $L(G)$) whenever the corresponding edges in G share a vertex (see [2]). The adjacency matrix $\mathbf{A}_L=\mathbf{A}(L(G))$ is given as follows:

$$\mathbf{A}_L=\mathbf{B}^{\mathrm{t}}\mathbf{B}-2\mathbf{I}_l, \tag{1}$$

where $l=|E(G)|$ and $\mathbf{B}=(b_{ij})$ is the incidence matrix of G: $b_{ij}=1$ if the edge e_i and the vertex v_j are incident, and $b_{ij}=0$ otherwise. Furthermore, we have

$$\mathbf{A}(G)={}^{\mathrm{t}}\mathbf{B}\mathbf{B}-\mathbf{D}. \tag{2}$$

The complexity of line graph $L(G)$ of an r-regular graph G is given as follows (see [2]):

Theorem 3. *Let G be a connected r-regular graph with n vertices and l edges. Then*

$$\kappa(L(G))=2^{l-n+1}r^{l-n-1}\kappa(G).$$

In Section 2, we treat zeta functions and complexities of semiregular bipartite graphs. As a corollary, we obtain the complexity of a complete bipartite graph. In Section 3, we give a decomposition formula for zeta function of a line graph of a semiregular bipartite graph. In Section 4, we present a formula for the complexity of a line graph of a semiregular bipartite graph. As an application, we can find the complexity of a line graph of a complete bipartite graph.

2. Zeta functions and complexities of semiregular bipartite graphs

A graph G is called *bipartite*, denoted by $G=(V_1, V_2)$ if there exists a partition $V(G)=V_1 \cup V_2$ of $V(G)$ such that $uv \in E(G)$ if and only if $u \in V_1$ and $v \in V_2$. A bipartite graph $G=(V_1, V_2)$ is called (q_1+1, q_2+1)-*semiregular* if $\deg_G v = q_i + 1$ for each $v \in V_i (i=1,2)$. For a (q_1+1, q_2+1)-semiregular bipartite graph $G=(V_1, V_2)$, let $G^{[i]}$ be the graph with vertex set V_i and an edge between two vertices in $G^{[i]}$ if there is a reduced path of length two between them in G for $i=1,2$. Then $G^{[1]}$ is $(q_1+1)q_2$-regular, and $G^{[2]}$ is $(q_2+1)q_1$-regular.

Hashimoto [3] treated multivariable zeta functions of bipartite graphs. For a graph G, let $\mathrm{Spec}(G)$ be the set of all eigenvalues of the adjacency matrix of G.

Theorem 4 (*Hashimoto [3]*). *Let $G=(V_1, V_2)$ be a connected (q_1+1, q_2+1)-semiregular bipartite graph with ν vertices and ε edges*, $|V_1|=n$ and $|V_2|=m (n \leqslant m)$. *Then*

$$\begin{aligned}\mathbf{Z}(G,u)^{-1} &= (1-u^2)^{\varepsilon-\nu}(1+q_2u^2)^{m-n}\prod_{j=1}^{n}(1-(\lambda_j^2-q_1-q_2)u^2+q_1q_2u^4)\\ &= (1-u^2)^{\varepsilon-\nu}(1+q_2u^2)^{m-n}\det(\mathbf{I}_n-(\mathbf{A}^{[1]}-(q_2-1)\mathbf{I}_n)u^2+q_1q_2u^4\mathbf{I}_n)\\ &= (1-u^2)^{\varepsilon-\nu}(1+q_1u^2)^{n-m}\det(\mathbf{I}_m-(\mathbf{A}^{[2]}-(q_1-1)\mathbf{I}_m)u^2+q_1q_2u^4\mathbf{I}_m),\end{aligned}$$

where $\mathrm{Spec}(G)=\{\pm\lambda_1,\ldots,\pm\lambda_n,0,\ldots,0\}$ *and* $\mathbf{A}^{[i]}=\mathbf{A}(G^{[i]})\ (i=1,2)$.

Note that

$$\mathbf{Z}(G,u)^{-1}=(1-u^2)^{\varepsilon-\nu}(1+q_2u^2)^{m-n}u^{2n}\Phi\left(G^{[1]};\frac{1+(q_2-1)u^2+q_1q_2u^4}{u^2}\right).$$

Next, we present a decomposition formula for the complexity of a semiregular bipartite graph G.

Theorem 5. *Let $G=(V_1, V_2)$ be a connected (q_1+1, q_2+1)-semiregular bipartite graph with ν vertices and ε edges, $|V_1|=n$ and $|V_2|=m (n \leqslant m)$. Then*

$$\begin{aligned}\kappa(G) &= \frac{1}{2(nq_1-m)}(q_2+1)^{m-n}\sum_{i=1}^{n}(4q_1q_2+2q_1+2q_2-2\lambda_i^2)\prod_{j\neq i}((q_1+1)(q_2+1)-\lambda_j^2)\\ &= \frac{q_1q_2-1}{nq_1-m}(q_2+1)^{m-n}\Phi'(G^{[1]};q_2(q_1+1)),\end{aligned}$$

where $\mathrm{Spec}(G)=\{\pm\lambda_1,\ldots,\pm\lambda_n,0,\ldots,0\}$.

Proof. Theorem 2 implies that

$$\kappa(G)=\frac{f'_G(1)}{2(|E(G)|-|V(G)|)}=\frac{f'_G(1)}{2(nq_1-m)}.$$

By Theorem 4, we have

$$f_G(u)=(1+q_2u^2)^{m-n}\det(\mathbf{I}_n-u^2(\mathbf{A}^{[1]}-q_2+1)+q_1q_2u^4\mathbf{I}_n).$$

Set

$$g(u)=\det(\mathbf{I}_n-u^2(\mathbf{A}^{[1]}-q_2+1)+q_1q_2u^4\mathbf{I}_n).$$

Since

$$g(1)=\det((q_1+1)q_2\mathbf{I}_n-\mathbf{A}^{[1]})=0, \tag{3}$$

we have

$$f'_G(1) = (1+q_2)^{m-n} g'(1).$$

Furthermore, by Theorem 4, we have

$$g(u) = \prod_{j=1}^{n} (1 - (\lambda_j^2 - q_1 - q_2)u^2 + q_1 q_2 u^4),$$

where $\mathrm{Spec}(G) = \{\pm\lambda_1, \ldots, \pm\lambda_n, 0, \ldots, 0\}$. Thus,

$$g'(1) = \sum_{i=1}^{n} (4q_1q_2 + 2q_1 + 2q_2 - 2\lambda_i^2) \prod_{j \neq i} ((q_1+1)(q_2+1) - \lambda_j^2).$$

Therefore, the first equation follows.

Now, let

$$h(u) = \frac{1 + (q_2 - 1)u^2 + q_1 q_2 u^4}{u^2}.$$

Then we have

$$g(u) = u^{2n} \Phi(G^{[1]}; h(u)).$$

Furthermore,

$$g'(u) = 2nu^{2n-1} \Phi(G^{[1]}; h(u)) + u^{2n} \Phi'(G^{[1]}; h(u)) h'(u).$$

By (3),

$$g'(1) = \Phi'(G^{[1]}; h(1)) h'(1) = \Phi'(G^{[1]}; q_2(q_1+1)) \cdot 2(q_1q_2 - 1).$$

Therefore, the second formula follows. □

From Theorem 5, we obtain a formula for the complexity of a complete bipartite graph $K_{m,m}$ (see [2]).

Corollary 1. *Let $K_{m,n}$ be a complete bipartite graph and $2 \leqslant n \leqslant m$. Then*

$$\kappa(K_{m,n}) = m^{n-1} n^{m-1}.$$

Proof. First, $K_{m,n}$ is a (m, n)-semiregular bipatite graph. Let V_1, V_2 be the partite sets of $K_{m,n}$, $|V_1| = n$ and $|V_2| = m$. Then we have $q_1 = m - 1$ and $q_2 = n - 1$. Furthermore,

$$\mathrm{Spec}(K_{m,n}) = \{\pm\sqrt{mn}, 0, \ldots, 0\}.$$

By Theorem 5, it follows that

$$\begin{aligned} \kappa(K_{m,n}) &= \frac{1}{2(mn-m-n)} n^{m-n} ((mn)^{n-1} (4(m-1)(n-1) + 2m - 2 + 2n - 2 - 2mn) + (n-1) \cdot 0) \\ &= \frac{1}{2(mn-m-n)} m^{n-1} n^{m-1} (2mn - 2m - 2n) = m^{n-1} n^{m-1}. \qquad \square \end{aligned}$$

3. Zeta functions of line graphs of semiregular bipartite graphs

Let G be a graph. Then the *line graph* $L(G)$ of G is the graph whose vertex set is the edge set $E(G)$ of G, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges in G have a vertex in common.

We express the zeta function of the line graph $L(G)$ for a semiregular bipartite graph G by using the characteristic polynomial of $G^{[1]}$.

Theorem 6. *Let $G=(V_1, V_2)$ be a connected (q_1+1, q_2+1)-semiregular bipartite graph with v vertices and ε edges, $|V_1|=n$ and $|V_2|=m(n\leqslant m)$. Suppose that $2\leqslant n\leqslant m$. Then*

$$\begin{aligned}
&\mathbf{Z}(L(G), u)^{-1}\\
&\quad=(1-u^2)^{n(q_1+1)(q_1+q_2-2)/2}(1+2u+(q_1+q_2-1)u^2)^{\varepsilon-v}(1+(1-q_2)u+(q_1+q_2-1)u^2)^{m-n}\\
&\qquad\times\prod_{i=1}^{n}(1+(2-q_1-q_2)u+(q_1q_2+q_1+q_2-1-\lambda_i^2)u^2\\
&\qquad+(q_1+q_2-1)(2-q_1-q_2)u^3+(q_1+q_2-1)^2u^4).\\
&\quad=(1-u^2)^{n(q_1+1)(q_1+q_2-2)/2}(1+2u+(q_1+q_2-1)u^2)^{\varepsilon-v}(1+(1-q_2)u+(q_1+q_2-1)u^2)^{m-n}u^{2n}\\
&\qquad\times\Phi\left(G^{[1]};\ \frac{1+(2-q_1-q_2)u+(q_1q_2+q_2-2)u^2+(q_1+q_2-1)(2-q_1-q_2)u^3+(q_1+q_2-1)^2u^4}{u^2}\right).
\end{aligned}$$

Proof. First, $L(G)$ is a (q_1+q_2)-regular graph. By Theorem 1, we have

$$\mathbf{Z}(L(G), u)^{-1}=(1-u^2)^{n(q_1+1)(q_1+q_2-2)/2}\det(\mathbf{I}_\varepsilon-u\mathbf{A}_L+u^2a\mathbf{I}_\varepsilon),$$

where $a=q_1+q_2-1$. But we have

$$\det(\mathbf{I}_\varepsilon-u\mathbf{A}_L+u^2a\mathbf{I}_\varepsilon)=u^\varepsilon\det\left(\frac{1+au^2}{u}\mathbf{I}_\varepsilon-\mathbf{A}_L\right).$$

Now, let

$$\mathbf{X}=\begin{bmatrix}\lambda\mathbf{I}_v & -{}^{\mathrm{t}}\mathbf{B}\\ \mathbf{0} & \mathbf{I}_\varepsilon\end{bmatrix},\quad \mathbf{Y}=\begin{bmatrix}\mathbf{I}_v & {}^{\mathrm{t}}\mathbf{B}\\ \mathbf{B} & \lambda\mathbf{I}_\varepsilon\end{bmatrix}.$$

Then we have

$$\mathbf{XY}=\begin{bmatrix}\lambda\mathbf{I}_v-{}^{\mathrm{t}}\mathbf{BB} & \mathbf{0}\\ \mathbf{B} & \lambda\mathbf{I}_\varepsilon\end{bmatrix}$$

and

$$\mathbf{YX}=\begin{bmatrix}\lambda\mathbf{I}_v & \mathbf{0}\\ \lambda\mathbf{B} & \lambda\mathbf{I}_\varepsilon-\mathbf{B}{}^{\mathrm{t}}\mathbf{B}\end{bmatrix}.$$

Since $\det XY=\det YX$,

$$\lambda^\varepsilon\det(\lambda\mathbf{I}_v-{}^{\mathrm{t}}\mathbf{BB})=\lambda^v\det(\lambda\mathbf{I}_\varepsilon-\mathbf{B}{}^{\mathrm{t}}\mathbf{B}),$$

i.e.,

$$\det(\lambda\mathbf{I}_\varepsilon-\mathbf{B}{}^{\mathrm{t}}\mathbf{B})=\lambda^{\varepsilon-v}\det(\lambda\mathbf{I}_v-{}^{\mathrm{t}}\mathbf{BB}).\tag{4}$$

By (1), (2) and (4), we have

$$\begin{aligned}
\det\left(\frac{1+au^2}{u}\mathbf{I}_\varepsilon-\mathbf{A}_L\right)&=\det\left(\frac{1+au^2}{u}\mathbf{I}_\varepsilon-\mathbf{B}{}^{\mathrm{t}}\mathbf{B}+2\mathbf{I}_\varepsilon\right)\\
&=\det\left(\frac{1+2u+au^2}{u}\mathbf{I}_\varepsilon-\mathbf{B}{}^{\mathrm{t}}\mathbf{B}\right)\\
&=\left(\frac{1+2u+au^2}{u}\right)^{\varepsilon-v}\det\left(\frac{1+2u+au^2}{u}\mathbf{I}_v-{}^{\mathrm{t}}\mathbf{BB}\right)\\
&=\left(\frac{1+2u+au^2}{u}\right)^{\varepsilon-v}\det\left(\frac{1+2u+au^2}{u}\mathbf{I}_v-\mathbf{A}(G)-\mathbf{D}\right).
\end{aligned}$$

Since G is bipartite, the vertices can be assumed to be ordered so that the adjacency matrix of G is

$$\mathbf{A}(G)=\begin{bmatrix}\mathbf{0} & \mathbf{E}\\ {}^{\mathrm{t}}\mathbf{E} & \mathbf{0}\end{bmatrix}.$$

Then $\mathbf{A}(G)$ is symmetric, and so there exists an $n\times n$ unitary matrix $\mathbf{W}$ such that

$$\mathbf{EW}=[\mathbf{F}\ \ \mathbf{0}]=\begin{bmatrix}\mu_1 & & 0 & 0 & \dots & 0\\ & \ddots & & \vdots & & \vdots\\ \star & & \mu_n & 0 & \dots & 0\end{bmatrix},$$

where $\mathbf{E}$ is an $n\times m$ matrix.

Now, let

$$\mathbf{P}=\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ \mathbf{0} & \mathbf{W}\end{bmatrix}.$$

Then we have

$${}^{\mathrm{t}}\bar{\mathbf{P}}\mathbf{A}(G)\mathbf{P}=\begin{bmatrix}\mathbf{0} & \mathbf{F} & \mathbf{0}\\ {}^{\mathrm{t}}\bar{\mathbf{F}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}.$$

Furthermore, since G is semiregular,

$${}^{\mathrm{t}}\bar{\mathbf{P}}\mathbf{D}\mathbf{P}=\mathbf{D}.$$

Thus,

$$\det\left(\frac{1+au^2}{u}\mathbf{I}_\varepsilon-\mathbf{A}_L\right)=\left(\frac{1+2u+au^2}{u}\right)^{\varepsilon-v}\left(\frac{1+\beta u+au^2}{u}\right)^{m-n}\times\det\begin{bmatrix}\dfrac{1+\alpha u+au^2}{u}\mathbf{I}_n & -\mathbf{F}\\ -{}^{\mathrm{t}}\bar{\mathbf{F}} & \dfrac{1+\beta u+au^2}{u}\mathbf{I}_n\end{bmatrix},$$

which, multiplying inside the determinant by $\begin{bmatrix}I_n & \\ 0 & I_n\end{bmatrix}$, equals

$$=\left(\frac{1+2u+au^2}{u}\right)^{\varepsilon-v}\left(\frac{1+\beta u+au^2}{u}\right)^{m-n}\det\begin{bmatrix}\dfrac{1+\alpha u+au^2}{u}\mathbf{I}_n & \mathbf{0}\\ -{}^{\mathrm{t}}\bar{\mathbf{F}} & \dfrac{1+\beta u+au^2}{u}\mathbf{I}_n-\dfrac{u}{1+\alpha u+au^2}{}^{\mathrm{t}}\bar{\mathbf{F}}\mathbf{F}\end{bmatrix}$$
$$=\left(\frac{1+2u+au^2}{u}\right)^{\varepsilon-v}\left(\frac{1+\beta u+au^2}{u}\right)^{m-n}\det\left(\frac{(1+\alpha u+au^2)(1+\beta u+au^2)}{u^2}\mathbf{I}_n-{}^{\mathrm{t}}\bar{\mathbf{F}}\mathbf{F}\right),$$

where $\alpha=1-q_1$ and $\beta=1-q_2$.

Since $\mathbf{A}(G)$ is symmetric, ${}^{\mathrm{t}}\bar{\mathbf{F}}\mathbf{F}$ is Hermitian and semipositive definite, i.e., the eigenvalues of ${}^{\mathrm{t}}\bar{\mathbf{F}}\mathbf{F}$ are of form:

$$\lambda_1^2,\dots,\lambda_n^2(\lambda_1,\dots,\lambda_n\geqslant 0).$$

Therefore it follows that

$$\begin{aligned}\mathbf{Z}(L(G),u)^{-1} &= \left(1-u^2\right)^{n(q_1+1)(q_1+q_2-2)/2}\left(\frac{1+2u+au^2}{u}\right)^{\varepsilon-\nu}\left(\frac{1+\beta u+au^2}{u}\right)^{m-n}u^{\varepsilon}\\ &\quad\times\prod_{i=1}^{n}\left(\frac{(1+\alpha u+au^2)(1+\beta u+au^2)}{u^2}-\lambda_i^2\right)\\ &= \left(1-u^2\right)^{n(q_1+1)(q_1+q_2-2)/2}(1+2u+au^2)^{\varepsilon-\nu}\left(1+\beta u+au^2\right)^{m-n}\\ &\quad\times\prod_{i=1}^{n}\left(\left(1+\alpha u+au^2\right)\left(1+\beta u+au^2\right)-\lambda_i^2u^2\right)\end{aligned}$$

from the first equation of Theorem 6 follows.

But we have

$$\det(t\mathbf{I}-\mathbf{A}(G)) = t^{m-n}\det(t^2\mathbf{I}-{}^{\mathrm{t}}\bar{\mathbf{F}}\mathbf{F}),$$

and so

$$\mathrm{Spec}(\mathbf{A}(G)) = \{\pm\lambda_1,\ldots,\pm\lambda_n,0,\ldots,0\}.$$

Thus, there exists a unitary matrix $\mathbf{S}$ such that

$${}^{\mathrm{t}}\bar{\mathbf{S}}\mathbf{A}(G)^2\mathbf{S} = \begin{bmatrix}\lambda_1^2 & & & & & 0\\ & \ddots & & & & \\ & & \lambda_n^2 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0\end{bmatrix},$$

where $\mathbf{S}$ is an $n\times n$ matrix. Furthermore, we have

$$\mathbf{A}(G)^2 = \mathbf{A}_2+\mathbf{D},$$

where $\mathbf{A}_2 = ((\mathbf{A}_2)_{uv})_{u,v\in V(G)}$: $(\mathbf{A}_2)_{uv}$ is the number of reduced (u,v)-paths with length 2. By the definition of $G^{[i]}(i=1,2)$,

$$\mathbf{A}(G)^2 = \begin{bmatrix}\mathbf{A}^{[1]}+(q_1+1)\mathbf{I}_n & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^{[2]}+(q_2+1)\mathbf{I}_m\end{bmatrix}.$$

Therefore, it follows that

$$\begin{aligned}&\Phi(G^{[1]};\ \frac{(1+\alpha u+au^2)(1+\beta u+au^2)}{u^2}-(q_1+1))\\ &\quad=\det(\frac{(1+\alpha u+au^2)(1+\beta u+au^2)}{u^2}\mathbf{I}_n-\mathbf{A}^{[1]}-(q_1+1)\mathbf{I}_n)\\ &\quad=\prod_{j=1}^{n}\left(\frac{(1+\alpha u+au^2)(1+\beta u+au^2)}{u^2}-\lambda_i^2\right).\end{aligned}$$

Hence, the second equation follows. □

For example, we consider the zeta function of the line graph of a complete bipartite graph. Let $K_{m,n}=(V_1,V_2)$ be the complete bipartite graph with $|V_1|=n$ and $|V_2|=m$ $(2\leqslant n\leqslant m)$. Then we have $q_1=m-1$, $q_2=n-1$, $\varepsilon=mn$

and $v = m + n$. By Theorem 6, we have

$$\begin{aligned}\mathbf{Z}(L(K_{m,n}), u)^{-1} = &(1-u^2)^{mn(m+n-4)/2}(1+2u+(m+n-3)u^2)^{mn-m-n}(1 + (2 - n)u + (m + n - 3)u^2)^{m-n}\\ &\times(1 - (m + n - 4)u + (mn - 2)u^2 - (m + n - 3)(m + n - 4)u^3 + (m + n - 3)^2u^4)^{n-1}\\ &\times (1 - (m + n - 4)u - 2u^2 - (m + n - 3)(m + n - 4)u^3 + (m + n - 3)^2u^4).\end{aligned}$$

4. Complexities of line graphs of semiregular bipartite graphs

We present a decomposition formula for the complexity of the line graph of a semiregular bipartite graph G.

Theorem 7. *Let $G = (V_1, V_2)$ be a connected $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph with v vertices and ε edges, $|V_1| = n$ and $|V_2| = m (n \leqslant m)$. Then*

$$\begin{aligned}\kappa(L(G)) &= \frac{1}{n(q_1 + q_2 - 2)}(q_1 + q_2 + 2)^{\varepsilon-v}(q_1 + 1)^{m-n-1}\\ &\quad\times \sum_{i=1}^{n}(q_1^2 + q_2^2 + 4q_1q_2 + 2q_1 + 2q_2 - 2 - 2\lambda_i^2)\prod_{j\neq i}((q_1 + 1)(q_2 + 1) - \lambda_j^2)\\ &= \frac{1}{n}(q_1 + q_2 + 2)^{\varepsilon-v+1}(q_1 + 1)^{m-n-1}\Phi'(G^{[1]}; q_2(q_1 + 1)),\end{aligned}$$

where $\mathrm{Spec}(\mathbf{A}(G)) = \{\pm\lambda_1, \ldots, \pm\lambda_n, 0, \ldots, 0\}$.

Proof. Theorem 2 implies that

$$\kappa(L(G)) = \frac{f'_{L(G)}(1)}{2(|E(L(G))| - |V(L(G))|)} = \frac{f'_{L(G)}(1)}{n(q_1 + 1)(q_1 + q_2 - 2)}.$$

By Theorem 6, we have

$$f_{L(G)}(u) = (1 + 2u + (q_1 + q_2 - 1)u^2)^{\varepsilon-v}(1 + (1 - q_2)u + (q_1 + q_2 - 1)u^2)^{m-n}u^{2n}\Phi(G^{[1]}; h_1(u)),$$

where

$$h_1(u) = \frac{(1 + \alpha u + au^2)(1 + \beta u + au^2)}{u^2} - q_1 - 1, \alpha = 1 - q_1, \beta = 1 - q_2, a = q_1 + q_2 - 1.$$

Since

$$\Phi(G^{[1]}; h_1(1)) = \det((q_1 + 1)q_2\mathbf{I}_n - \mathbf{A}^{[1]}) = 0, \tag{5}$$

we have

$$f'_{L(G)}(1) = (q_1 + q_2 + 2)^{\varepsilon-v}(1 + q_1)^{m-n}\{u^{2n}\Phi(G^{[1]}; h_1(u))\}'|_{u=1}.$$

But, we have

$$u^{2n}\Phi(G^{[1]}; h_1(u)) = \prod_{j=1}^{n}((1 + \alpha u + au^2)(1 + \beta u + au^2) - \lambda_i^2u^2),$$

where $\mathrm{Spec}(G) = \{\pm\lambda_1, \ldots, \pm\lambda_n, 0, \ldots, 0\}$. Thus,

$$\{u^{2n}\Phi(G^{[1]}; h_1(u))\}'|_{u=1} = \sum_{i=1}^{n}(q_1^2 + q_2^2 + 4q_1q_2 + 2q_1 + 2q_2 - 2 - 2\lambda_i^2)\prod_{j\neq i}((q_1 + 1)(q_2 + 1) - \lambda_j^2).$$

Furthermore, by (5), we have

$$\begin{aligned} f'_{L(G)}(1) &= (q_1+q_2+2)^{\varepsilon-v}(1+q_1)^{m-n}\Phi'(G^{[1]}; h_1(1))h'_1(1) \\ &= (q_1+q_2+2)^{\varepsilon-v}(1+q_1)^{m-n}\Phi'(G^{[1]}; q_2(q_1+1))(q_1+q_2+2)(q_1+q_2-2). \end{aligned}$$

Therefore, the result follows. $\square$

By Theorems 5 and 7, the complexity of the line graph $L(G)$ of a semiregular bipartite graph G is expressed by that of G. This is an analogue of Theorem 3 for a semiregular bipartite graph.

Corollary 2. *Let $G=(V_1, V_2)$ be a connected (q_1+1, q_2+1)-semiregular bipartite graph with v vertices and ε edges, $|V_1|=n$ and $|V_2|=m (n \leqslant m)$. Then*

$$\kappa(L(G)) = \frac{nq_1-m}{n(q_1+1)}(q_1+q_2+2)^{\varepsilon-v+1}\frac{1}{q_1q_2-1}\left(\frac{q_1+1}{q_2+1}\right)^{m-n}\kappa(G).$$

By Corollaries 1 and 2, we obtain a formula for the complexity of the line graph $L(K_{m,n})$ of a complete bipartite graph $K_{m,m}$.

Corollary 3. *Let $K_{m,n}$ be a complete bipartite graph and $2 \leqslant n \leqslant m$. Then*

$$\kappa(L(K_{m,n})) = (m+n)^{(m-1)(n-1)}m^{m-2}n^{n-2}.$$

Proof. At first, $K_{m,n}$ is a (m, n)-semiregular bipatite graph. Let V_1, V_2 be the partite set of $K_{m,n}$, $|V_1|=n$ and $|V_2|=m$. Then we have $q_1=m-1$,$q_2=n-1$,$v=m+n$ and $\varepsilon=mn$. By Corollaries 1,2, it follows that

$$\begin{aligned} \kappa(L(K_{m,n})) &= \frac{mn-m-n}{mn}(m+n)^{mn-m-n+1}\frac{1}{mn-m-n}\left(\frac{m}{n}\right)^{m-n}m^{n-1}n^{m-1} \\ &= (m+n)^{mn-m-n+1}m^{m-2}n^{n-2}. \qquad \square \end{aligned}$$

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